$\ast$ $E = \{1, x, x^2\} \subseteq P_2(\mathbb{R})$

$$L(a + bx + cx^2) = (a+b+c) + (a+b)x + (a+b)x^2$$

Compute $\mathrm{Rep}_{E,E}(L)$.

$\hookrightarrow \mathrm{Rep}_{B,D}(L)$

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ \cup\!\vert & & \cup\!\vert \\ B & & D \end{array}$$

$$\mathrm{Rep}_{B,D}(L) = \Big[[L(b_1)]_D \Big| [L(b_2)]_D \Big| \cdots \Big| [L(b_n)]_D \Big]$$

Want $\mathrm{Rep}_{E,E}(L) = \Big[[L(1)]_E \Big| [L(x)]_E \Big| [L(x^2)]_E \Big]$

NB: $L(1) = L(1 \cdot 1 + 0 \cdot x + 0 \cdot x^2) = (1+0+0) + (1+0)x + (1+0)x^2$
$= 1 \cdot 1 + 1 \cdot x + 1 \cdot x^2$

$\ast$ $L(a + bx + cx^2) = (a+b+c) + (a+b)x + (a+b)x^2$

$L(x) = L(0 \cdot 1 + 1 \cdot x + 0 \cdot x^2) = 1 + x + x^2$

$L(x^2) = L(0 \cdot 1 + 0 \cdot x + 1 \cdot x^2) = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$$[L(1)]_E = \begin{array}{cl} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{array}{l} \text{coeff of } b_1 = 1 \\ \text{coeff of } b_2 = x \\ \text{coeff of } b_3 = x^2 \end{array} \end{array}$$

$\ast$

$$[L(x)]_E = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad [L(x^2)]_E = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathrm{Rep}_{E,E}(L) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Remark: this was easy because E was a very nice basis...

In general, we need to use:

$$\begin{tikzcd}
V_B \arrow[r, "\mathrm{Rep}_{B,D}(L)"] \arrow[d, "\mathrm{Rep}_{B,B'}(id)"'] & W_D \arrow[d, "\mathrm{Rep}_{D,D'}(id)"] \\
V_{B'} \arrow[r, "\mathrm{Rep}_{B',D'}(L)"'] & W_{D'}
\end{tikzcd}$$

These matrices describe coord changes.

$$\mathrm{Rep}_{B,D}(L) = \left[\mathrm{Rep}_{D',D}(id)\right] \mathrm{Rep}_{B',D'}(L) \cdot \left[\mathrm{Rep}_{B,B'}(id)\right]$$

$$\begin{tikzcd}
P_2(\mathbb{R})_B \arrow[r, "\boxed{\mathrm{Rep}_{B,D}(L)}"] \arrow[d, "\mathrm{Rep}_{B,E}(id)"'] & P_2(\mathbb{R})_D \arrow[d, "\mathrm{Rep}_{D,E}(id)"] \\
P_2(\mathbb{R})_E \arrow[r, "\mathrm{Rep}_{E,E}(L)"'] & P_2(\mathbb{R})_E
\end{tikzcd}$$

$$\mathrm{Rep}_{E,E}(L) = \begin{bmatrix} | & | & | \\ | & | & 0 \\ & & 0 \end{bmatrix} \leftarrow \text{was easy to compute}$$

Orthogonal Complement: $W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 : \begin{array}{l} 2a - b + c = 0 \\ a - b - d = 0 \end{array} \right\}$

$$W = \mathrm{Col}(A) \implies W^{\perp} = \mathrm{null}(A^T)$$

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{l} \boxed{b = 2a + c} \\ d = a - b \end{array} \right\} \qquad d = a - b = a - (2a + c)$$

$$= \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{l} b = 2a + c \\ d = -a - c \end{array} \right\} = \left\{ \begin{bmatrix} a \\ 2a + c \\ c \\ -a - c \end{bmatrix} : a, c \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} a \\ 2a \\ 0 \\ -a \end{bmatrix} + \begin{bmatrix} 0 \\ c \\ c \\ -c \end{bmatrix} : a, c \in \mathbb{R} \right\}$$

$$= \left\{ a \underline{\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}} + c \underline{\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}} : a, c \in \mathbb{R} \right\}$$

$$= \mathrm{Col} \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}} = \mathrm{Col}(A)$$

$$\therefore W^{\perp} = \mathrm{null}(A^T)$$

$$= \mathrm{null} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$= \mathrm{null} \begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \\ \;\; a \quad\; b \quad\; c \quad\; d \end{array}$$

$$= \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{l} a - 2c + d = 0 \\ b + c - d = 0 \end{array} \right\}$$

$\hookrightarrow$ <u>Keep going</u>... give a basis...

$\boxtimes$

$$\boxed{W = \mathrm{Col}(A) \implies W^{\perp} = \mathrm{null}(A^T).}$$

<u>Ex</u>: Show $B = \{1+x, 1+2x+x^2, x+2x\}$ spans $\underline{P_2(\mathbb{R})}$.

($x+2x = 3x$)

<u>Sol</u>: $$[B]_E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix} = M$$

$$\mathrm{RREF}(M) = \mathrm{RREF}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} = \mathrm{RREF}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} = \mathrm{RREF}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; !! \; \smile$$

So because $\mathrm{rank}(M) = \mathrm{rank}(\mathrm{RREF}(M)) = 3$, and $\dim(P_2(\mathbb{R})) = 3$, B spans. $\boxtimes$

Ex: Apply Gram-Schmitt process to $v_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix}$.

Sol: $u_1 = v_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$

$$u_2 = v_2 - \boxed{\text{proj}_{u_1}(v_2)} = v_2 - \boxed{\frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1}$$

$$= \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix} - \frac{\begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix}}{\begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}} \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix} - \frac{-15 - 9 - 18 - 3}{9 + 1 + 4 + 1} \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix} + \frac{3(5 + 3 + 6 + 1)}{15} \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ -3 \\ 0 \end{pmatrix}$$

$\therefore$ $u_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 4 \\ 6 \\ -3 \\ 0 \end{pmatrix}$ span same space as v_1, v_2

and they are orthogonal :) ☒

$$P = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}$$

$\lambda = 2$ $\lambda = 1$ $\lambda = -5$ $\lambda = \pi$

To make orthonormal:

Nxt step: Normalize each vector $\rightsquigarrow$ $u_1 \rightsquigarrow w_1 = \frac{1}{|u_1|} u_1$, $u_2 \rightsquigarrow w_2 = \frac{1}{|u_2|} u_2$

In the previous example: $u_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 4 \\ 6 \\ -3 \\ 0 \end{pmatrix}$

$$|u_1| = \sqrt{3^2+1^2+2^2+1^2} = \sqrt{15}, \quad |u_2| = \sqrt{4^2+6^2+3^2} = \sqrt{5^2+6^2} = \sqrt{61}$$

$$\therefore \quad w_1 = \frac{1}{|u_1|}u_1 = \frac{1}{\sqrt{15}}\begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}, \quad w_2 = \frac{1}{|u_2|}u_2 = \frac{1}{\sqrt{61}}\begin{pmatrix} 4 \\ 6 \\ -3 \\ 0 \end{pmatrix}$$

is orthonormal collection spanning same space as v_1, v_2. $\boxed{\checkmark}$

$$\begin{tikzcd} V_B \arrow[r, "\mathrm{Rep}_{B,D}(L)"] \arrow[d, "\sim"'] & W_D \arrow[d, "\sim"] \\ \mathbb{R}^{\dim(V)} \arrow[r, "M = \mathrm{Rep}_{B,D}(L)"'] & \mathbb{R}^{\dim(W)} \end{tikzcd}$$

$L: V \to W$ isomorphism iff L^{-1} exists.

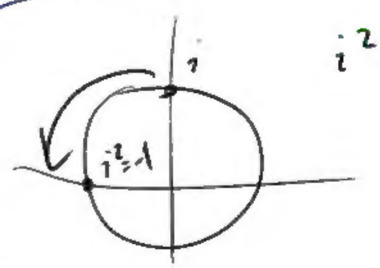
$* \; L(B) = X$ — basis of W iff L is an iso.

$* \; L^{-1}: W \to V$ via $L^{-1}(L(b)) = b$.

extend linearly (using lin. ext prop. from before Exam 2).

Ex: $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has e-values $\lambda = \pm i$ (check).

is NOT diagonalizable over $\mathbb{R}$,
but IS diagonalizable over $\mathbb{C}$



$i^2 = -1$

$\mathbb{C}$

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a + b = 0 \right\}$$